A Primer on Pseudoholomorphic Curves

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1. Some Complex Stuff!

1.1. **Almost Complex Manifolds.** Let M be a manifold. Over M we have the tangent bundle, TM. We can then take the *endomorhpism bundle* of TM, written $\mathrm{End}(TM)$, which is again a vector bundle over M. The fiber over $x \in M$ consists of the endomorphisms of the tangent space T_xM. What we are interested in are particular kinds of sections of this bundle.

Definition (Almost Complex Structure)**.** *An* almost complex structure *is a section,* $J : M \to \mathrm{End}(TM)$*, of* $\mathrm{End}(TM)$ *such that* $J^2 = -\mathrm{id}$ *(here, we think of* J *as a family of maps* $J_x : T_xM \to T_xM$*,* $x \in M$*).*
A pair (M, J) *is called an almost complex manifold.*

Perhaps a more intuitive way to think of an almost complex structure is akin to the way you can think of a Riemannian metric. One definiton of a Riemannian metric is an inner product on the tangent spaces of M, which varies smoothly across the manifold. You can think of an almost complex structre as a linear complex structure on each tangent space which varies smoothly over the manifold. Some interesting consequences of M admitting an almost complex structure are that it is necessarily even dimensional and orientable. The fact that it is even dimensional should be easy enough to see from the fact that each tangent space has a complex structure, and thus must be even dimensional. The adjective "almost" refers to the non-integrability of the section J. Let's say some more about this:

Recall that, $i \in \mathbb{C}$, thought of as an endomorphism of a complex vector space V has two eigenvalues: 1 and -1, and so two eigenspaces, which we call $V^{(1,0)}$ and $V^{(0,1)}$. An almost complex structure J on a manifold M will allow us to make a similar splitting of the complexified tangent bundle of M: $TM^{\mathbb{C}} := TM \otimes \mathbb{C}$ (here, $\mathbb{C}$ is to be thought of as the trivial complex line bundle over M) into two subbundles, $T^{(1,0)}M$ and $T^{(0,1)}M$. We are already at one way of telling if J is integrable:

$$J \text{ is integrable } \iff [T^{(1,0)}M, T^{(1,0)}M] \subset T^{(1,0)}M$$

that is, if the Lie bracket of any two sections of $T^{(1,0)}M$ is again a section of $T^{(1,0)}M$ (sections of this are (holomorphic) vector fields). This looks like what the word "integrable" means in the sense of *Frobenius' Theorem*. Perhaps the quickest way to check whether J is integrable is with the *Nieulander-Nirenberg Theorem*, which states that J is integrable if and only if $N_J = 0$, where N_J is the *Nijenhuis tensor* which is defined as

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

for vector fields X, Y on M.

Yet another way to think of this is as follows. We can begin constructing differential forms on M using $TM^{\mathbb{C}}$, and so we can construct what are called (p,q)-forms. First, some notation:

$$T^{(p,q)}M = \underbrace{T^{(1,0)}M \otimes \cdots \otimes T^{(1,0)}M}_{p-\text{times}} \otimes \underbrace{T^{(0,1)}M \otimes \cdots \otimes T^{(0,1)}M}_{q-\text{times}}$$

Definition ((p,q)-form)**.** *Let (M,J) be an almost complex manifold. We define the space of* (p,q)-forms *on M as*

$$\Omega^{(p,q)}(M) := \Gamma((T^{(p,q)}M)^*)$$

This gives us a way to decompose the usual differential r-forms on M (where, again, we've used the complexified tangent bundle):

$$\Omega^r(M)^{\mathbb{C}} = \sum_{p+q=r} \Omega^{(p,q)}(M).$$

On $\Omega^\bullet(M)^{\mathbb{C}}$, we have the usual deRham differential

$$d : \Omega^r(M)^{\mathbb{C}} \to \Omega^{r+1}(M)^{\mathbb{C}}$$

Since $\Omega^r(M)^{\mathbb{C}}$ is written as a direct sum, we have projection maps

$$\pi_{(p,q)} : \Omega^r(M)^{\mathbb{C}} \to \Omega^{(p,q)}(M)$$

for any $p+q=r$, and this allows us to define two new operators

$$\begin{aligned}
\partial &: \Omega^{(p,q)}(M) \to \Omega^{(p+1,q)}(M) \\
\partial &= \pi_{(p+1,q)} \circ d \\
\bar{\partial} &: \Omega^{(p,q)}(M) \to \Omega^{(p,q+1)}(M) \\
\bar{\partial} &= \pi_{(p,q+1)} \circ d
\end{aligned}$$

Now, the sum of all the projection maps $\pi_{(p,q)}$, $p+q=r$, must be the identity map on $\Omega^r(M)^{\mathbb{C}}$, and as such, we get that

$$d = \sum_{r+s=p+q+1} \pi_{(r,s)} \circ d = \partial + \bar{\partial} + \cdots .$$

This leads us to another definiton of J being integrable:

$$J \text{ is integrable} \iff d = \partial + \bar{\partial},$$

and with some work, one also has the additional equivalent condition:

$$J \text{ is integrable} \iff \bar{\partial}^2 = 0.$$

We usually call an *integrable* almost complex structure simply a *complex structure*. A manifold together with a complex structure is called a *complex manifold*.

Remark.

(1) *This isn't exactly the definiton of a complex manifold, but it implies that the manifold is complex. The definition of a complex manifold is a manifold for which the transition maps are holomorphic. The precise story is easier to say in coordinates. A complex*

manifold comes with local holomorphic coordinates $(z_1, ..., z_n)$, $z_k = x_k + iy_k$, *around any point* $p \in M$ *such that*

$$i\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k} \quad \text{and} \quad i\frac{\partial}{\partial y_k} = -\frac{\partial}{\partial x_k}.$$

Replacing i *with* J, *we can define a section* J *of* $\mathrm{End}(TM)$, *which is an integrable almost complex structure. On an almost complex manifold, around any point* p, *we can find local coordinates* $(x_1, y_1, ..., x_n, y_n)$ *for which*

$$J\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k} \quad \text{and} \quad J\frac{\partial}{\partial y_k} = -\frac{\partial}{\partial x_k}$$

at p, *but we cannot guarantee that this even holds in a neighborhood of* p. *If, however, around any point we can find a coordinate neighborhood in which the above holds for any point in the neighborhood, we can patch the structures together to form a complex structure. This is yet another (rather impractical) equivalent definiton of an almost complex structure being integrable.*

(2) *On a complex manifold* $\bar{\partial}^2 = 0$, *and so we get a chain complex* $\left(\Omega^{p,\bullet}(M), \bar{\partial}\right)$, *and so we can take its cohomology. This is called the* Dolbeault cohomology of M.

(3) *It can be shown that an almost complex structure on a* 2*-dimensional manifold is always integrable.*

1.2. **Riemann Surfaces.**

Definition (Riemann Surface)**.** *A* Riemann surface *is a one dimensional complex manifold. (Keep in mind, this is one* complex *dimensional!!) That is, locally, Riemann surfaces look like* $\mathbb{C}$ *(with it's complex structure).*

Example.

(1) *The sphere,* $\mathbb{CP}^1$.
(2) $\mathbb{C}$ *itself.*
(3) *Any open subset of* $\mathbb{C}$.
(4) Σ_g, *the surface of genus* g. *Note that these, for* $g > 0$, *have several distinct complex structures. These are the objects in the Teichmüller space (or, at least one interpretation of them) of the surface* Σ_g.
(5) *One can take the sphere* $\mathbb{CP}^1$ *and puncture it any number of times.*

2. Pseudoholomorphic Curves

Let (M, J) be an almost complex manifold, and let (Σ, j) be a Riemann surface (j is the complex structure on Σ).

Definition (Pseudoholomorphic Curve)**.** *A smooth map* $u : (\Sigma, j) \to (M, J)$ *is called* pseudoholomorphic *or* (j, J)-holomorphic*, or simply* J-holomorphic *if* j *is understood, if* du *is complex-linear with respect to* j *and* J, *i.e.,*

$$du \circ j = J \circ du,$$

which can alternatively be written

$$du + J \circ du \circ j = 0.$$

We define the operator $\bar{\partial}_J$ which picks out the complex anti-linear part of du by

$$\bar{\partial}_J(u) = \frac{1}{2}\left(du + J \circ du \circ j\right)$$

then we can say a curve u is (j,J)-holomorphic if

$$\bar{\partial}_J(u) = 0.$$

This is the analogue of the Cauchy-Riemann equations for J-holomorphic curves. Let's see that this makes sense with our usual notion of holomorphic on $\mathbb{C}^n$:

Let's first start with passing to local coordinates on Σ. We can work in a chart $\phi_\alpha : U_\alpha \to \mathbb{C}$ on Σ ($U_\alpha \subset \Sigma$ is open). By doing this, we can assume that our Riemann surface is $(\mathbb{C}, i)$, where i is the usual complex structure. Let's give $\mathbb{C}$ the coordinates $z = s + it$. Define $u_\alpha = u \circ \phi_\alpha^{-1}$. In this case we have

$$\begin{aligned} \bar{\partial}_J u_\alpha &= \frac{1}{2}\left[\left(\frac{\partial u_\alpha}{\partial s}ds + \frac{\partial u_\alpha}{\partial t}dt\right) + J(u_\alpha)\left(\frac{\partial u_\alpha}{\partial t}ds - \frac{\partial u_\alpha}{\partial s}dt\right)\right] \\ &= \frac{1}{2}\left[\left(\frac{\partial u_\alpha}{\partial s} + J(u_\alpha)\frac{\partial u_\alpha}{\partial t}\right)ds + \left(\frac{\partial u_\alpha}{\partial t} - J(u_\alpha)\frac{\partial u_\alpha}{\partial s}\right)dt\right] \end{aligned}$$

From this, we can see that $\bar{\partial}_J u_\alpha = 0$ if

$$\frac{\partial u_\alpha}{\partial s} + J(u_\alpha)\frac{\partial u_\alpha}{\partial t} = 0 \tag{1}$$

(the dt coefficient is this, multiplied by $J(u_\alpha)$).

Now, if we assume $M = \mathbb{C}^n$ with the usual complex structure i, under the identification $\mathbb{C}^n \cong \mathbb{R}^n \oplus i\mathbb{R}^n$ we get

$$i = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Letting $u_\alpha = f + ig$, equation (1) becomes

$$\left(\frac{\partial f}{\partial s} + i\frac{\partial g}{\partial s}\right) + i\left(\frac{\partial f}{\partial t} + i\frac{\partial g}{\partial t}\right) = \left(\frac{\partial f}{\partial s} - \frac{\partial g}{\partial t}\right) + i\left(\frac{\partial f}{\partial t} + \frac{\partial g}{\partial s}\right) = 0,$$

the familiar Cauchy-Riemann equations (if you like, take $n = 1$).

3. Symplectic Manifolds

3.1. Definitions and Examples. Let M be a smooth manifold and let $\omega \in \Omega^2(M)$. ω is called *closed* if $d\omega = 0$. ω is called *nondegenerate* if one of these equivalent things hold

- the induced bundle map
$$\tilde{\omega} : TM \to T^*M$$
is a bundle isomorphism.
- given a vector field X on M, if for all vector fields Y on M
$$\omega(X, Y) \equiv 0,$$
then $X = 0$.
- $\omega^n \neq 0$, where $\dim M = 2n$.

Definition (Symplectic Manifold). *A 2-form ω on M which is closed and nondegenerate is called a symplectic form on M. A pair (M, ω) is called a symplectic manifold.*

Remark. *A symplectic manifold is necessarily even dimensional (real dimension).*

Example.

(1) $\mathbb{C}^n(z_1, ..., z_n)$ *with* $\omega_{st} = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$. *If you write* $z_j = x_j + iy_j$, *then* $\omega_{st} = \sum_{k=1}^{n} dx_j \wedge dy_j$.

(2) $\mathbb{CP}^n$ *with the Fubini-Study form*

$$\omega_{FS} = i\partial\bar{\partial}\log(|z|^2)$$

where $z = [z_0 : \cdots : z_n]$ *are homogeneous coordinates on* $\mathbb{CP}^n$.

(3) *The* $2n$*-torus* $\mathbb{T}^{2n} = S^1(p_1) \times S^1(q_1) \times \cdots \times S^1(p_n) \times S^1(q_n)$ *with* $\omega = \sum_{j=1}^{n} dp_j \wedge dq_j$.

There are two special types of submanifolds of symplectic manifolds we will be concerned with here:

Definition.

(1) *A submanifold* $L \subset M$ *such that* $\omega|_{TL} \equiv 0$ *is called a* Lagrangian *submanifold. Usually we just call these Lagrangians. A Lagrangian submanifold necessarily has half the dimension of* M.

(2) *A submanifold* $S \subset M$ *such that* $\omega|_{TS}$ *is again a symplectic form is called a* symplectic *submanifold. A symplectic submanifold can have any even codimension.*

Example.

(1) *The* n*-torus* $\mathbb{T}^n = S^1 \times \cdots \times S^1 \subset \mathbb{C} \times \cdots \times \mathbb{C} = \mathbb{C}^n$ *where* $\mathbb{C}^n$ *has the standard symplectic form is Lagrangian. Additionally, the usual inclusion* $\mathbb{R}^n \subset \mathbb{C}^n$ *is Lagrangian. The submanifold* $\mathbb{RP}^n \subset \mathbb{CP}^n$ *is Lagrangian with respect to the Fubini-Study form.*

(2) $\mathbb{C}^m \subset \mathbb{C}^n$ *for* $m < n$ *are symplectic submanifolds. The submanifolds* $\mathbb{CP}^m \subset \mathbb{CP}^n$, $m < n$ *are symplectic submanifolds.*

3.2. Symplectic Manifolds and Almost Complex Structures.

Let (M, ω) be a symplectic manifold. An almost complex structure J on M is called

- ω*-tame*, or is said to be *tamed by* ω, if

$$\omega(v, Jv) > 0$$

- ω*-compatible* if it is ω-tame and in addition satisfies

$$\omega(Jv, Jw) = \omega(v, w)$$

Remark. *Note that, if* J *is* ω*-compatible, we get an induced Riemannian metric on the symplectic manifold*

$$g_J(v, w) = \omega(v, Jw).$$

Proposition. *Given a symplectic manifold* (M, ω), *there exists an* ω*-compatible almost complex structure on* M. *In fact, in the space* $\mathcal{J}$ *of almost complex structures on* M, *the set of* ω*-compatible almost complex structures,* $\mathcal{J}(\omega)$, *is contractible.*

The proof of this fact is via a simple lemma

Lemma. *On a manifold M, a symplectic form ω, a Riemannian metric g, and an almost complex structure J which are compatible. Then any two determine the third, i.e.,*

- *given ω and an ω-compatible J, we get a Riemannian metric g_J as above*
- *given g and J, we get a symplectic form ω_J which tames J by*

$$\omega_J(v, w) = g(Jv, w)$$

- *given g and ω, we get an ω-compatible J by*

$$J(v) = \tilde{g}^{-1} \circ \tilde{\omega}(v)$$

where $\tilde{g}, \tilde{\omega} : TM \to T^*M$ are the induced isomorphisms.

This lemma has to do with the so-called "2 out of 3 property" of $U(n)$:

$$U(n) = O(2n) \cap GL(n, \mathbb{C}) \cap Sp(2n, \mathbb{R})$$

moreover, $U(n)$ is the intersection of any two of these.

Proof of theorem. The fact that every smooth manifold has a Riemannian metric is a basic exercise in differential topology. By the third point of the lemma, we thus get an ω-compatible almost complex structure. The fact that $\mathcal{J}(\omega)$ is contractible follows from the fact that the space of Riemannian metrics is contractible. This follows from the fact that the space of inner products on a vector space is affine. $\square$

3.3. **Hamiltonian diffeomorphisms.**

Let (M, ω) be a symplectic manifold. Given a smooth function $h : M \to \mathbb{R}$, define the *Hamiltonian vector field of h* to be the vector field X_h such that

$$\imath_{X_h}\omega = dh.$$

A *Hamiltonian diffeomorphism* of M is defined to be the time 1 flow, ψ, of a Hamiltonian vector field.

4. Theorems and Applications

4.1. **Generalization of the Riemann Mapping Theorem.**

Consider again the symplectic manifold $(\mathbb{C}^n, \omega_{st})$. Let D denote the unit disc in $\mathbb{C}$. The proof of this result is an application of holomorphic curves, but is quite involved.

Theorem (Gromov '85)**.** *Let $L \subset \mathbb{C}^n$ be a compact Lagrangian submanifold. Then there exists a nonconstant holomorphic disc $u : D \to \mathbb{C}^n$ such that $u(\partial D) \subset L$.*

Corollary. *A Lagrangian as above has $H^1(L; \mathbb{R}) \neq 0$.*

Proof. Let $\lambda = \frac{1}{2}\sum_{j=1}^n (x_j dy_j - y_j dx_j)$. (Notice that $d\lambda = \omega_{st}$.) Then the integral of λ around $C = u(\partial D)$ is nonzero, and so $[\lambda] \neq 0$. This is because

$$\int_C \lambda = \int_{u(D)} d\lambda = \int_D u^*\omega = \int_D |du|^2 \neq 0.$$

$\square$

Another corollary of this theorem essentially says that there are always intersections between a Lagrangian submanifold and any Hamiltonian deformation of it (under appropriate assumptions).

Definition (Convex at Infinity)**.** *A noncompact symplectic manifold* (M, ω) *is called convex at infinity if there exists a pair* (f, J)*, where* J *is an* ω*-compatible almost complex structure and* $f : M \to [0, \infty)$ *is a proper smooth function such that*

$$\omega_f(v, Jv) \geq 0, \qquad \omega_f := -d(df \circ J),$$

for every x *outside some compact subset of* M *and every* $v \in T_x M$*.*

Corollary. *Let* (M, ω) *be a symplectic manifold without boundary, and assume that* (M, ω) *is convex at infinity. Let* $L \subset M$ *be a compact Lagrangian submanifold such that* $[\omega]$ *vansishes on* $\pi_2(M, L)$*. Let* $\psi : M \to M$ *be a Hamiltonian symplectomorphism. Then* $\psi(M) \cap M \neq \varnothing$*.*

4.2. **The Nonsqueezing Theorem.** Let $B^{2n}(r)$ be the closed ball of radius r and center 0 in $\mathbb{R}^{2n}$. Another application of holomorphic curves is the following

Theorem (Gromov)**.** *If* $\iota : B^{2n}(r) \to \mathbb{R}^{2n}$ *is a symplectic embedding (the image is a symplectic submanifold of* $\mathbb{R}^{2n}$*) such that* $\iota(B^{2n}(r)) \subset B^2(R) \times \mathbb{R}^{2n-2}$*, then* $r \leq R$

and a further generalization of it is

Theorem. *Let* (M, ω) *be a compact symplectic manifold of dimension* $2n - 2$ *such that* $\pi_2(M) = 0$*. If there is a symplectic embedding of the ball* $(B^{2n}(r), \omega_{st})$ *into* $B^2(R) \times M$*, then* $r \leq R$*.*

4.3. **Classification of Compact Symplectic 4-manifolds.**

Theorem (The Classification Theorem (Gromov '85 [1]; McDuff '90 [2]) (*))**.** *Let* (V, ω) *be a compact symplectic* 4*-manifold and suppose* (V, C, ω) *is a minimal pair where* C *is a symplectically embedded* 2*-sphere with self-intersection* $C \cdot C \geq 0$*. Then* (V, ω) *is symplectomorphic either to* $\mathbb{CP}^2$ *with its usual Kähler form, or to a symplectic* S^2*-bundle over a Riemann surface* M*. Further, this symplectomorphism may be chosen so that it takes* C *either to a complex line or quadric in* $\mathbb{CP}^2$*, or to a fiber of the* S^2*-bundle, or, if* $M = S^2$*, to a section of this bundle.*

References

[1] Mikhail Gromov, *Pseudo holomorphic curves in symplectic manifolds.* Invent. math. 82, 307-347, 1985.

[2] Dusa McDuff, *The structure of Rational and Ruled Symplectic* 4*-Manifolds*, Journal of the AMS, vol. 3, No. 3 (1990), 679-712.

[3] Dusa McDuff and Dietmar Salamon, *J-holomorphic Curves and Symplectic Topology.* Second edition, AMS Colloquium Publications, vol. 52 (2012).